

QUADRATIC PACKING POLYNOMIALS ON SECTORS OF \mathbb{R}^2 **Madeline Brandt**¹*Reed College, Portland, OR 97202, United States*
mbrandt@reed.edu**Abstract**

A polynomial $p(x, y)$ on a region S in the plane is called a packing polynomial if the restriction of $p(x, y)$ to $S \cap \mathbb{Z}^2$ yields a bijection to \mathbb{N} . In this paper, we determine all quadratic packing polynomials on rational sectors of \mathbb{R}^2 .

1. Introduction

Let $S \subseteq \mathbb{R}^2$, and let $I = S \cap \mathbb{N}^2$. A polynomial $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a *packing polynomial* on S if $f|_I$ is a bijection from I to \mathbb{N} . In 1923 Fueter and Pólya [1] proved that the *Cantor polynomials*,

$$f(x, y) = \frac{(x+y)^2}{2} + \frac{x+3y}{2} \quad \text{and} \quad g(x, y) = \frac{(x+y)^2}{2} + \frac{3x+y}{2},$$

are the only quadratic packing polynomials on $\mathbb{R}_{\geq 0}^2$, and Vsemirnov [5] gives two elementary proofs of this theorem. Fueter and Pólya also conjectured that the Cantor polynomials are in fact the only packing polynomials on \mathbb{N}^2 . In 1978, Lew and Rosenberg [2] showed that there are no cubic or quartic packing polynomials on \mathbb{N}^2 , but the existence of higher degree packing polynomials remains unknown.

In this paper, we study quadratic packing polynomials on rational sectors. For all $\alpha \in \mathbb{R}_{\geq 0}$, let

$$S(\alpha) = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } y \leq \alpha x\},$$

and let $I(\alpha)$ be the set of lattice points contained in $S(\alpha)$. If α is an integer (rational, irrational), we call $S(\alpha)$ an integral (rational, irrational) sector. The following results are known for quadratic packing polynomials on rational sectors.

In 2013, Nathanson [3] gave two quadratic packing polynomials on $S(n)$, for $n \in \mathbb{N}$,

$$f_n(x, y) = \frac{n}{2}x^2 + \left(1 - \frac{n}{2}\right)x + y,$$

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$$g_n(x, y) = \frac{n}{2}x^2 + \left(1 + \frac{n}{2}\right)x - y.$$

Subsequently, Stanton [4] proved that these polynomials, along with four polynomials on $S(3)$ and $S(4)$, are the only quadratic packing polynomials on integral sectors. After classifying the polynomials on integral sectors, Stanton discovered a necessary condition for quadratic packing polynomials on rational sectors.

Theorem 1 (Stanton [4]). *Let $n/m \geq 1$ and $(n, m) = 1$. Suppose $S(\frac{n}{m})$ has a quadratic packing polynomial p , and let $p_2(x, y)$ denote the homogeneous quadratic part of p . Then n divides $(m-1)^2$, and*

$$p_2(x, y) = \frac{n}{2} \left(x - \frac{m-1}{n}y \right)^2.$$

We observe that the restriction $n/m \geq 1$ does not result in any loss of generality because there is a bijection, observed by Nathanson in [3], from $I(n/m)$ to $I\left(\frac{n}{m-n\lfloor m/n \rfloor}\right)$ given by

$$W_{n/m} = \begin{pmatrix} 1 & -\lfloor m/n \rfloor \\ 0 & 1 \end{pmatrix}.$$

In light of this, we will say that two packing polynomials p on $S(\alpha)$ and q on $S(\beta)$ are *equivalent* if there exists a linear map $T : I(\alpha) \rightarrow I(\beta)$ which is a bijection from $I(\alpha)$ to $I(\beta)$ such that $p = q \circ T$.

In this paper, we determine all quadratic packing polynomials on rational sectors up to equivalence by finding the necessary equations for quadratic packing polynomials on rational sectors, and then by finding a sufficient condition for the resulting polynomials to be packing polynomials. In Section 2, we start by introducing the notion of a k -stair polynomial, giving some basic results on their properties, and demonstrating that all quadratic packing polynomials must be k -stair polynomials. We proceed to give necessary and sufficient conditions for k -stair polynomials to be packing polynomials in Section 3. We conclude in Section 4 with our main result: the classification of all quadratic packing polynomials on rational sectors.

2. k -Stair Polynomials

For the remainder of this paper, assume that m and n are relatively prime, the integer n divides $(m-1)^2$, and let $l = (n, m-1)$. Let $p(x, y)$ be a packing polynomial, so that by Theorem 1 we may write

$$p(x, y) = \frac{n}{2} \left(x - \frac{m-1}{n}y \right)^2 + dx + ey + f.$$

Definition We call the line segment

$$y = \frac{n}{m-1}x - c\frac{l}{m-1}$$

for $c \in \mathbb{N}$ and $(x, y) \in S(n/m)$ the c^{th} staircase of $I(n/m)$. A *stair* is a point with integer coordinates on a staircase. The *first* stair on the c^{th} staircase is the stair with minimal x -coordinate. Two stairs r, s are *consecutive* if they are on the same staircase and there is no other stair on the line segment from r to s . For $c \in \mathbb{N}$, define

$$S_c \equiv \left\{ (x, y) \in I\left(\frac{n}{m}\right) \mid y = \frac{n}{m-1}x - c\frac{l}{m-1} \right\}.$$

Lemma 1. We have $I(n/m) = \cup_{c \in \mathbb{N}} S_c$.

Proof. Clearly $I(n/m) \supseteq \cup_{c \in \mathbb{N}} S_c$. For the other direction, let $(a, b) \in I(\frac{n}{m})$, and let $h = a\frac{n}{l} - b\frac{m-1}{l}$. Consider the following line with slope $\frac{n}{m-1}$ through the point (a, b) :

$$y = \frac{n}{m-1}x - \frac{l}{m-1}h.$$

Since $l \mid n$ and $l \mid m-1$, and $b/a \leq n/m$, we have $h \in \mathbb{N}$. Therefore, (a, b) is a stair on S_h . \square

Lemma 2. If p is a quadratic packing polynomial on $S(\frac{n}{m})$, and $(x, y) \in S(\frac{n}{m})$, then for some $k \in \mathbb{N}$,

$$p\left(x + \frac{m-1}{l}, y + \frac{n}{l}\right) - p(x, y) = \pm k.$$

Proof. By Stanton's necessary condition, $p_2(x, y) = \frac{n}{2}(x - \frac{m-1}{n}y)^2$. If L is a staircase, then $p|_L$ is linear because $p_2|_L$ is constant. \square

Definition Let $p : S(\frac{n}{m}) \rightarrow \mathbb{R}$ be a quadratic polynomial with $p_2(x, y) = \frac{n}{2}(x - \frac{m-1}{n}y)^2$ and $p(\mathbb{Z}^2) \subseteq \mathbb{Z}$. Then p is a k -stair polynomial if for any two consecutive stairs r, s , we have $p(r) - p(s) = \pm k$. If $|r| < |s|$ and $p(s) - p(r) = k$, then p we call *ascending*, otherwise we call p *descending*.

Lemma 2 shows that all quadratic packing polynomials on sectors $S(n/m)$ are k -stair polynomials for some k . Figure 1 gives examples of two 1-stair packing polynomials. The next proposition shows that ascending and descending k stair packing polynomials are equivalent.

Proposition 1. There is an ascending k -stair packing polynomial on $S(\frac{n}{m})$ if and only if there is a descending k -stair packing polynomial on $S(\frac{n}{n+2-m})$.

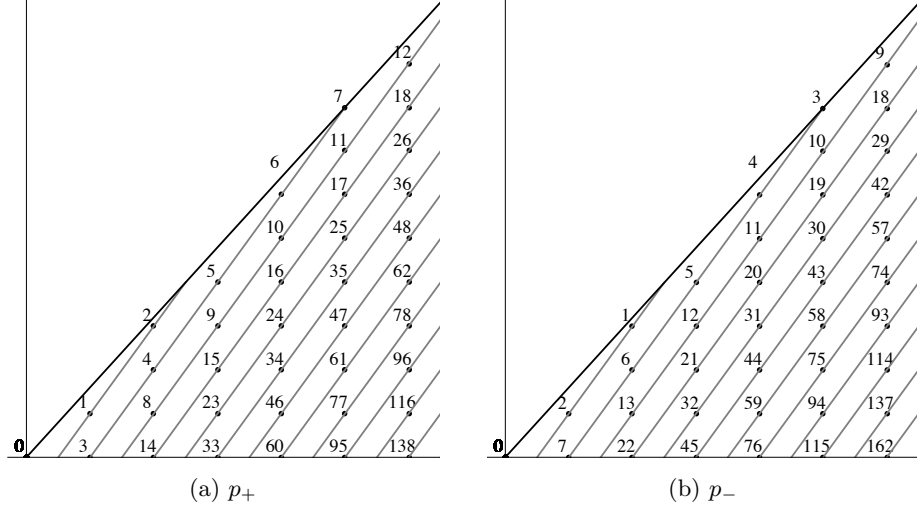


Figure 1: An ascending 1-stair packing polynomial p_+ and a descending 1-stair packing polynomial p_- , both on $S(\frac{8}{5})$.

Proof. Let $m' = n + 2 - m$, and

$$T_{n/m} = \begin{pmatrix} m' & \frac{1-m'm}{n} \\ n & -m \end{pmatrix}.$$

It is straightforward to show that $T_{n/m}$ is a bijection from $I(\frac{n}{m})$ to $I(\frac{n}{m'})$, so that if p is a quadratic packing polynomial on $S(\frac{n}{m})$, then $p \circ T_{n/m'}$ is a quadratic packing polynomial on $S(\frac{n}{m'})$. A simple calculation shows that if p is an ascending (descending) k -stair polynomial, then $p \circ T_{n/m'}$ is a descending (ascending) k -stair packing polynomial on $S(\frac{n}{m'})$. \square

2.1. Properties of k -stair polynomials.

Let p be a k -stair polynomial. Then the following immediate observations can be made. If a, b lie on the same staircase, then $p(a) \equiv p(b) \pmod{k}$. Moreover, the numbers $0, 1, \dots, k-1$ must all occur on the first (last) stairs for an ascending (descending) k -stair packing polynomial, because otherwise the first (last) stairs will take on negative values. Figure 2 gives an example of a 3-stair packing polynomial. The following lemma provides more information about the behavior of k -stair polynomials.

Lemma 3. *There exists some j_0 such that whenever $j \geq j_0$, if $(a, b) \in S_j$, and $(a', b') \in S_{j+k}$, then*

$$p(a, b) \equiv p(a', b') \pmod{k}.$$

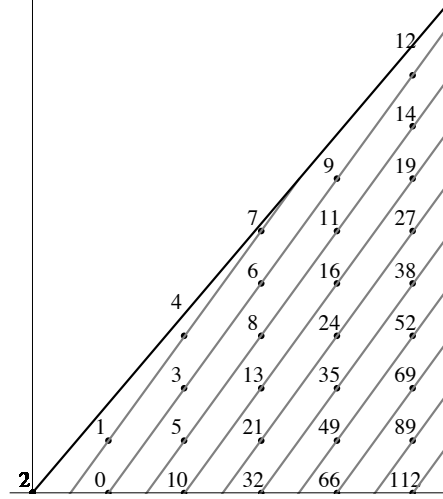


Figure 2: An ascending 3-stair polynomial on $S(12/7)$.

Proof. The function $p(x, 0)$ is increasing for all $x > x_0$ for some x_0 . Let $j_0 = x_0 \frac{n}{l}$, and $j \geq j_0$. Suppose that for any $(a, b) \in S_j$, we have that $p(a, b) \equiv c \pmod k$. Let $j' > j$ be the smallest integer such that for any $(a', b') \in S_{j'}$, we have that $p(a', b') \equiv c \pmod k$.

Suppose $j' - j > k$. Then by the pigeonhole principle, there exist i, i' such that for any $(a, b) \in S_i$ and any $(a', b') \in S_{i'}$, $p(a, b) \equiv p(a', b') \pmod k$. Let $s(l)$ be the number of stairs on the l^{th} staircase, and let $\bar{p}(l)$ be the value of p on the first stair on the l^{th} staircase. Note that $s(i) = \lfloor (m-1)i \frac{l}{n} \rfloor \geq \lfloor (m-1)(j \frac{l}{n} + \frac{l}{n}) \rfloor \geq \lfloor (m-1)j \frac{l}{n} \rfloor + 1$.

Then,

$$\begin{aligned}
 p(j'l/n, 0) &\leq \bar{p}(j') \\
 &= \bar{p}(j) + k \cdot s(j) \\
 &< \bar{p}(i) + k \cdot s(j) \\
 &\leq \bar{p}(i) + k \cdot (s(i) - 1) \\
 &= \bar{p}(i') - k \\
 &< p(i'l/n, 0),
 \end{aligned}$$

which is a contradiction because $p(x, 0)$ is strictly increasing for $x > x_0$. Therefore, $j' - j \leq k$. If $j' - j < k$, then there exist i, i' such that $i' - i > k$ and for any $(a, b) \in S_i$ and any $(a', b') \in S_{i'}$, we have $p(a, b) \equiv p(a', b') \pmod k$, but we previously showed that this can not happen. Therefore, $j' - j = k$. \square

Lemma 4. *Let p be an ascending k -stair packing polynomial.*

1. If j is a (large enough) integer, and (a, b) is the first stair on the $(j\frac{n}{l} + k)^{th}$ staircase, then

$$p(a, b) - k = p(mj, nj).$$

2. If (a, b) is the first stair on the j^{th} staircase (for large enough j), and (c, d) is the intersection of the line $y = \frac{n}{m}x$ and the $j - k^{th}$ staircase, then

$$p(a, b) - k \leq p(c, d).$$

Proof.

1. This is an immediate consequence of the bijectivity of p and Lemma 3.
2. If $p(a, b) - k > p(c, d)$ then the value $p(a, b) - k$ does not occur on any i^{th} staircase where $i \equiv j \pmod k$. However, if $p(x, y) \equiv p(a, b) \pmod k$, and $p(x, y)$ is on the i^{th} staircase, then $i \equiv j \pmod k$, by Lemma 3. Therefore, the value $p(a, b) - k$ is missing from the range of p , so p is not surjective.

□

Lemma 5. Let r be the multiplicative inverse of $\frac{m-1}{l} \pmod{\frac{n}{l}}$. Let $j \equiv j' \pmod{\frac{n}{l}}$ and $z = -j'r \pmod{\frac{n}{l}}$. Then first stair on the j^{th} staircase has coordinates

$$\left(\frac{m-1}{n}z + j\frac{l}{n}, z \right).$$

Proof. Let $\phi : \mathbb{Z}_{\frac{n}{l}} \rightarrow \mathbb{Z}_{\frac{n}{l}}$ send $j \in \mathbb{Z}_{\frac{n}{l}}$ to the y coordinate of the first stair of the j^{th} staircase. We only define ϕ on the first stairs because if $j \equiv i \pmod{\frac{n}{l}}$, then the y coordinate of the first stair of the j^{th} staircase will be the same as the y coordinate of the first stair of the i^{th} staircase. Note that $\phi(j) \in \mathbb{Z}_{n/l}$ because if (x, y) is the first stair on the j^{th} staircase and $\phi(j) > \frac{n}{l}$, then $(x - \frac{m-1}{l}, y - \frac{n}{l})$ is a stair on the j^{th} staircase with smaller x -coordinate.

Then observe that $\phi^{-1}(-i) = i\frac{m-1}{l}$, so that $\phi(j) = -j(\frac{m-1}{l})^{-1}$. The x coordinate comes from solving

$$y = \frac{n}{m-1} \left(x - j\frac{l}{n} \right).$$

□

3. Necessary and Sufficient Conditions for k -Stair Packing Polynomials on $S(\frac{n}{m})$

Theorem 2. Let $p(x, y) = \frac{n}{2}(x - \frac{m-1}{n}y)^2 + dx + ey + f$ be a packing polynomial on $S(n/m)$, where $l = (n, m-1)$. Then either

1. p is an ascending k -stair polynomial where $k \equiv \frac{m-1}{l} \pmod{\frac{n}{l}}$, and

$$p(x, y) = \frac{n}{2} \left(x - \frac{m-1}{n} y \right)^2 + \left(1 - \frac{kl}{2} \right) x + \frac{2(1-m) + kl(m+1)}{2n} y + f,$$

or

2. p is a descending k -stair polynomial where $k \equiv -\frac{m-1}{l} \pmod{\frac{n}{l}}$, and

$$p(x, y) = \frac{n}{2} \left(x - \frac{m-1}{n} y \right)^2 + \left(1 + \frac{kl}{2} \right) x + \frac{2(1-m) - kl(m+1)}{2n} y + f.$$

Proof. Let p be an ascending k -stair packing polynomial on $S(n/m)$. Let $k = q\frac{n}{l} + k'$, where $k' < \frac{n}{l}$ and q is an integer; let r be the multiplicative inverse of $\frac{m-1}{l} \pmod{\frac{n}{l}}$, and let $z = -k'r \pmod{\frac{n}{l}}$. By Lemma 2 and Lemma 4, for large enough j and x , we have

$$k = p \left(x + \frac{(m-1)}{l}, y + \frac{n}{l} \right) - p(x, y)$$

and

$$k = p \left(\frac{m-1}{n} z + j + k \frac{l}{n}, z \right) - p(mj, nj),$$

so that

$$e = \frac{-2(m-1)n + ((m+1)nq + 2(m-1)z)l + k'(m+1)l^2}{2nl},$$

and

$$d = \frac{1}{2} \left(-nq - 2z + \frac{2n}{l} - k'l \right).$$

By Lemma 4, we also have when (a, b) is a first stair (for large enough a),

$$p \left(a - \frac{(m-1)}{l}, b - \frac{n}{l} \right) \leq p \left(m \left(b \frac{(1-m)}{n} + a - \left(q\frac{n}{l} + k' \right) \frac{l}{n} \right), n \left(b \frac{(1-m)}{n} + a - \left(q\frac{n}{l} + k' \right) \frac{l}{n} \right) \right).$$

Plugging in these points using the e and d given above, we find that this inequality is satisfied if and only if

$$0 \leq -\frac{(b-z)(nq + k'l)}{n}.$$

In particular, there are first stairs (a, b) where $b = \frac{n}{l} - 1$, and because $\frac{nq+k'l}{n} > 0$, we therefore must have that $0 \leq z - b$. Since $z < \frac{n}{l}$, this implies that $z = \frac{n}{l} - 1$. So, $-k'r \equiv -1$, so $k' \equiv r^{-1}$, which implies that $k' = \frac{m-1}{l} \pmod{\frac{n}{l}}$. The coefficients of $p(x, y)$ follow from simplifying d and e with this requirement.

The case where p is a descending polynomial follows from Proposition 1. \square

Since the coefficients e, f must be set to satisfy the inequalities from Lemma 4 for large enough x, y , they will also satisfy the inequalities for all other x, y . Therefore, we find that $p(x, y)$ automatically satisfies the inequalities from Lemma 4 for all x, y .

Theorem 3. *Let a_1, \dots, a_k be the first stairs on the first k staircases on $S(n/m)$. Then p is an ascending packing polynomial if and only if p is a k -stair polynomial of the necessary form given in Theorem 2, and*

$$\{p(a_1), \dots, p(a_k)\} = \{0, 1, \dots, k-1\}.$$

Proof. Suppose p is an ascending k -stair polynomial with the necessary form, and $\{p(a_1), \dots, p(a_k)\} = \{0, 1, \dots, k-1\}$. For any $i \in \{1, \dots, k\}$, let

$$R_i = \cup \{S_c \mid c \equiv i \pmod{k}\}.$$

If $p|_{R_i}$ is a bijection from $R_i \cap \mathbb{N}^2$ to $p(a_i) + k\mathbb{N}$ for any i , then p is a packing polynomial on $S(n/m)$.

Since p satisfies the inequality from Lemma 4, $p|_{R_i}$ is surjective to $p(a_i) + k\mathbb{N}$ (since no values congruent to $i \pmod{k}$ will be skipped). Then $p|_{R_i}$ will be injective if whenever (a, b) is the first stair on the j^{th} staircase and (c, d) is the last stair on the $(j-k)^{\text{th}}$ staircase, we have

$$p(a, b) > p(c, d).$$

We also have

$$0 < \frac{m-1+nq}{n} = p\left(j\frac{l}{n}, 0\right) - p\left(m\frac{l}{n}(j-k), l(j-k)\right).$$

Since

$$p(a, b) \geq p\left(j\frac{l}{n}, 0\right) > p\left(m\frac{l}{n}(j-k), l(j-k)\right) \geq p(c, d),$$

we have $p(a, b) > p(c, d)$, so $p|_{R_i}$ is injective and so p is a packing polynomial.

Conversely, suppose p is a packing polynomial. Let $i \in \{0, 1, \dots, k-1\}$. If $p(a_i) \equiv j \pmod{k}$ where $0 \leq j < k$, but $p(a_i) \neq j$, then by the above, for any $a \in R_i$, we have $p(a) \geq p(a_i)$. So, there is no (x, y) such that $p(x, y) = j$. Therefore, $p(a_i) = j$ for some $j \in \{0, 1, \dots, k-1\}$. On the other hand, if there is some $j \in \{0, 1, \dots, k-1\}$ such that $p(a_i) \neq j$ for any $i \in \{1, \dots, k\}$, then p will never achieve values congruent to $j \pmod{k}$. Therefore, $\{p(a_1), \dots, p(a_k)\} = \{0, 1, \dots, k-1\}$.

□

4. Packing Polynomials on $S(\frac{n}{m})$

Now we are prepared to determine, up to isomorphism, the k -stair packing polynomials for each k . In particular, we will prove that there are no k -stair polynomials when $k \geq 4$. We first provide two additional results.

Proposition 2. *If there is an ascending k -stair packing polynomial on $S(\frac{n}{m})$ for $\frac{n}{m} > 1$ and $m \neq 1$, then $k \mid l$.*

Proof. Suppose $l \not\equiv 0 \pmod{k}$. Let (a_i, b_i) be the first stair on the i^{th} staircase, where $i < k$. Let $\xi_i = -i(\frac{m-1}{l})^{-1} \pmod{\frac{n}{l}}$. Then by Lemma 5 and Theorem 2 we have

$$p(a_i, b_i) - f = \frac{l}{2n}(i(i-k)l + 2(i + k\xi_i)).$$

By Theorem 3, for any $i, j < k$, we need that $p(a_i, b_i) \not\equiv p(a_j, b_j) \pmod{k}$. Since $\frac{n}{l}$ and $k = q\frac{n}{l} + \frac{m-1}{l}$ (for some integer q) are relatively prime, n/l is not a zero divisor in \mathbb{Z}_k . Therefore, $p(a_i, b_i) \not\equiv p(a_j, b_j) \pmod{k}$ if and only if $\frac{n}{l}p(a_i, b_i) \not\equiv \frac{n}{l}p(a_j, b_j) \pmod{k}$. Then,

$$\begin{aligned} \frac{n}{l}p(a_i, b_i) &= \frac{l}{2}i^2 + i\left(1 - \frac{kl}{2}\right) \\ &= \frac{l}{2}\left(i + \frac{1}{l} - \frac{k}{2}\right)^2 - \frac{l}{2}\left(\frac{k}{2} - \frac{1}{l}\right)^2. \end{aligned}$$

So, if $j = -i - \frac{2}{l} + k$, then $p(a_i, b_i) \equiv p(a_j, b_j) \pmod{k}$. If $j = i$, then $i = \frac{k}{l} - \frac{1}{l}$, which will only happen for one i , so for some i, j , we have that $p(a_i, b_i) \equiv p(a_j, b_j) \pmod{k}$. \square

Theorem 4. *If p is a k -stair packing polynomial on $S(\frac{n}{m})$ where $\frac{n}{m} > 1$ and $m \neq 1$, then either $k = \frac{m-1}{l}$ or $\frac{n}{m} = \frac{12}{7}$.*

Proof. By Theorem 2, we have $k = q\frac{n}{l} + \frac{m-1}{l}$ for some $q \in \mathbb{N}$. Suppose that $q \neq 0$. Then the point $(1, 0)$ is the first stair on the $\frac{n}{l}^{\text{th}}$ staircase, and substituting $l = q\frac{n}{k} + \frac{m-1}{k}$, we find that

$$p(1, 0) - f = \frac{1}{2}(3 - m + n - nq).$$

By Theorem 3, we have $|p(1, 0) - f| \leq k - 1$. In particular, the inequality $p(1, 0) - f \geq -(k - 1)$ implies that

$$2k \geq m - 1 + n(q - 1).$$

If $q > 1$, then

$$\begin{aligned} 2k &\geq m - 1 + n(q - 1) \\ &\geq m - 1 + n \\ &> 2(m - 1), \end{aligned}$$

which implies that $k > m - 1$. This is impossible; by Proposition 2, we have $k \mid l$, and $l \mid m - 1$, so $k \leq m - 1$ (when $m \neq 0$).

Suppose that $q = 1$. Then $2k \geq m - 1 \geq k$, and since $k \mid m - 1$, we must have $2k = m - 1$, or $k = m - 1 = l$.

1. Suppose that $2k = m - 1$. Then $k \mid l$ and $l \mid 2k$, so either $l = 2k = m - 1$ or $l = k$.

- (a) Suppose that $l = 2k = m - 1$. Then the equality $k = q \frac{n}{l} + \frac{m-1}{l}$ implies that $k = \frac{n}{m-1} + 1$, so

$$\frac{(m-1)^2}{n} = \frac{m-1}{k-1} = \frac{2k}{k-1}.$$

Moreover, this quantity is an integer. We conclude that either $k = 2$ or $(k-1) \mid 2$, so that $k = 2$ or 3 .

If $k = 2$, then $m = 5$ and $n = 4$, contradicting the assumption that $n > m$.

If $k = 3$, then $m = 7$ and $n = 12$, and we obtain a 3-stair packing polynomial on $S(\frac{12}{7})$.

- (b) Suppose that $2l = 2k = m - 1$. Then the equality $k = q \frac{n}{l} + \frac{m-1}{l}$ implies that $k = \frac{2n}{m-1} + 2$, so

$$\frac{(m-1)^2}{n} = \frac{2(m-1)}{k-2} = \frac{4k}{k-2}.$$

Moreover, this quantity is an integer. For any integer k , $\gcd(k, k-2) = 1$ or 2 .

If $\gcd(k, k-2) = 1$, then $(k-2) \mid 4$, which forces $k = 3$. If $k = 3$, then $n = 3$ and $m = 7$, contradicting the assumption that $n > m$.

If $\gcd(k, k-2) = 2$, then $2 \mid k$, so $\frac{n}{m-1}$ is an integer, forcing $l = m-1 = 2l$, so this case cannot occur.

2. Suppose that $k = m - 1 = l$. Then the equality $k = q \frac{n}{l} + \frac{m-1}{l}$ implies that $k = \frac{n}{m-1} + 1$, so

$$\frac{(m-1)^2}{n} = \frac{m-1}{k-1} = \frac{k}{k-1}.$$

Moreover, this quantity is an integer, so that $k = 2$. It follows that $m = 3$ and $n = 2$, contradicting the assumption that $n > m$.

Therefore, if $k \neq \frac{m-1}{n}$, then $\frac{n}{m} = \frac{12}{7}$.

□

Theorem 5. Let $\frac{n}{m} \in \mathbb{Q}$, $(n, m) = 1$, $m \neq 1$, and $\frac{n}{m} > 1$. The following results give the k -stair packing polynomials on sectors $S(\frac{n}{m})$ for $k \in \{1, 2, 3, 4\}$.

1. There is an ascending 1-stair packing polynomial p on $S(n/m)$ if and only if $n \mid (m-1)^2$ and $m-1 \mid n$.
2. There is an ascending 2-stair packing polynomial p on $S(n/m)$ if and only if $m \equiv 9 \pmod{16}$ and $n = \frac{1}{16}(m-1)^2$.
3. There is an ascending 3-stair packing polynomial p on $S(n/m)$ if and only if $m \equiv 10 \pmod{27}$ or $m \equiv 19 \pmod{27}$ and $n = \frac{1}{27}(m-1)^2$, or $\frac{n}{m} = \frac{12}{7}$.
4. There are no 4-stair packing polynomials.

Proof. 1. By Theorem 2, we have $k = 1$ if and only if $\frac{m-1}{l} = 1$, so $m-1 \mid n$. By setting $f = 0$, the sufficient condition from Theorem 3 is satisfied since $p(0, 0) = f = 0$.

2. Suppose p is an ascending 2-stair polynomial on $S(n/m)$. By Theorem 4, $k = \frac{m-1}{l} = 2$, $2 \nmid \frac{n}{l}$. Note that $2(\frac{n}{l} - \frac{n/l-1}{2}) \equiv 1 \pmod{\frac{n}{l}}$, so the first stair on the first staircase (by Lemma 5) is

$$\left(1, \frac{n/l-1}{2}\right).$$

Then by Theorem 3, p is a packing polynomial if and only if

$$\left\{p(0, 0), p\left(1, \frac{n/l-1}{2}\right)\right\} = \{0, 1\}.$$

Since $p(0, 0) = f$, we may find an f which satisfies this as long as $|p(0, 0) - p(1, \frac{n/l-1}{2})| = 1$. So, using the necessary form of p given in Theorem 2 we have,

$$\begin{aligned} \pm 1 &= p\left(1, \frac{n/l-1}{2}\right) - p(0, 0) \\ &= \frac{(m-1)^2 - 8n}{8n}. \end{aligned}$$

Since $m \neq 1$, we find that p is a packing polynomial if and only if $n = \frac{(m-1)^2}{16}$. Then because $l = \frac{m-1}{2}$, we have that $8 \mid m-1$ but $16 \nmid m-1$.

In the case where $m = 1$, Stanton in [4] found two 2-stair packing polynomials on $S(4)$. However, by the above we note that $S(4/9)$ has a 2-stair packing polynomial, and Stanton's polynomials are both equivalent to the ascending 2-stair packing polynomial on $S(4/9)$.

3. By Theorem 4, either $n/m = 12/7$, or $k = 3 = \frac{m-1}{l}$. The case where $3 = \frac{m-1}{l}$ follows by a method similar to that used to prove (2). We note that the ascending 3-stair packing polynomial on $S(12/19)$ is equivalent to the ascending 3-stair packing polynomial on $S(12/7)$.

In the case where $m = 1$, Stanton in [4] found two 3-stair packing polynomials on $S(3)$. In a fashion similar to (2), these are both equivalent to the ascending 3-stair packing polynomial on $S(3/10)$.

4. Stanton [4] proved that there are no 4-stair packing polynomials on $S(n)$. By Theorem 4, we have that if $m \neq 1$, then $4 = \frac{m-1}{l}$. Since $\frac{n}{l}$ is relatively prime to $\frac{m-1}{l} = 4$, either $\frac{n}{l} \equiv 1 \pmod{k}$ or $\frac{n}{l} \equiv 3 \pmod{k}$.

- (a) Suppose $\frac{n}{l} \equiv 1 \pmod{k}$. Then if (a_i, b_i) are the first stairs on the first 4 staircases by Lemma 5 we have

$$\begin{aligned} (a_0, b_0) &= (0, 0) \\ (a_1, b_1) &= \left(1, \frac{n/l - 1}{4}\right) \\ (a_2, b_2) &= \left(2, \frac{n/l - 1}{2}\right) \\ (a_3, b_3) &= \left(3, 3\frac{n/l - 1}{4}\right), \end{aligned}$$

and so

$$\begin{aligned} p(a_0, b_0) - f &= 0 \\ p(a_1, b_1) - f &= \frac{-3(m-1)^2 + 32n}{32n} \\ p(a_2, b_2) - f &= \frac{(m-1)^2 - 16n}{8n} \\ p(a_3, b_3) - f &= \frac{-3((m-1)^2 - 32n)}{32n}. \end{aligned}$$

By Theorem 3, $\{p(a_i, b_i) - f \mid i \in \{0, 1, 2, 3\}\} \subset \{-3, -2, -1, 0, 1, 2, 3\}$. Let $x = p(a_1, b_1) - f$. Then $32n(1 - x) = 3(m-1)^2$, so $x < 0$. Let $x' = p(a_2, b_2) - f$. Then $-8nx' + 16n = (m-1)^2$, so

$$8n(2 - x') = \frac{32n}{3}(1 - x),$$

which implies that $3 \mid 1 - x$ and $4 \mid 2 - x'$, so $x = x' = -2$. We conclude that p is not injective.

- (b) The case where $\frac{n}{l} \equiv 3 \pmod{k}$ follows similarly; we find that there are no packing polynomials in this case.

□

Theorem 6. *There are no k -stair polynomials for $k \geq 4$.*

Proof. Stanton [4] showed that in the case where $m = 1$, there are no k -stair packing polynomials for $k \geq 4$, so assume $m \neq 1$. Let p be a k -stair packing polynomial where $k \geq 4$. By Theorem 4, this implies that $k = \frac{m-1}{l}$. Then by Theorem 2,

$$p(x, y) = \frac{n}{2} \left(x - \frac{m-1}{n} y \right)^2 + \frac{3-m}{2} x + \frac{(m-1)^2}{2n} y + f.$$

Let (a, b) be the first stair on S_1 . By Theorem 3, we know that

$$|p(a, b) - f| \leq k - 1.$$

Also, $p(a, b) - p(\frac{l}{n}, 0) \leq k - 1$ since p is k -stair, so that $p(\frac{l}{n}, 0) - f \geq -2(k - 1)$. From the above form of $p(x, y)$, we find that

$$p\left(\frac{l}{n}, 0\right) = \frac{(m-1)^2}{2kn} \left(\frac{1-k}{k} + \frac{2}{m-1} \right).$$

Then $p(\frac{l}{n}, 0) - f \geq -2(k - 1)$ if and only if

$$\frac{(m-1)^2}{kn} \left(\frac{1}{k} - \frac{2}{(m-1)(k-1)} \right) \leq 4.$$

Now, we claim that $n \leq \frac{l^2}{k}$. Suppose that n and $m - 1$ are divisible by k^j , and no higher power of k divides $m - 1$. Since $m - 1 = kl$, we have that k^{j-1} is the highest power of k that divides l . But, since n and $m - 1$ are both divisible by k^j , we also have $k^j \mid l$, which is a contradiction. Therefore, a higher power of k divides $m - 1$ than divides n . Also, $n \mid (m - 1)^2$, so $n \mid l^2$. If k^i is the highest power of k that divides l and $k^{2i} \mid n$, then by the above, $2i < i + 1$, which implies that $i = 0$. This is a contradiction since $k \mid l$. Therefore, $n \mid \frac{l^2}{k}$.

Plugging in $n = \frac{l^2}{k}$, we find that

$$k - \frac{2k}{l(k-1)} \leq \frac{(m-1)^2}{kn} \left(\frac{1}{k} - \frac{2}{(m-1)(k-1)} \right) \leq 4. \quad (1)$$

Therefore, if

$$k > 4 + \frac{2k}{l(k-1)},$$

then inequality (1) does not hold, and so p is not a packing polynomial. We find that $k > 4 + \frac{2k}{l(k-1)}$ whenever $k \geq 5$. This result, along with Theorem 5, implies that there are no k -stair packing polynomials when $k \geq 4$. □

Thus, there are only k -stair polynomials for $k \in \{1, 2, 3\}$. By Theorem 5 and extra details provided in the proof, we conclude that up to equivalence, the polynomials given in Theorem 2.1 on the sectors $S(\frac{n}{m})$ given by Theorem 5 along with Nathanson's f_n, g_n on $S(n)$ represent all quadratic packing polynomials on rational sectors.

5. Future Directions

Lew and Rosenberg [2] proved that there are no packing polynomials of degree three or four on \mathbb{N}^2 . It is an open question whether there exist packing polynomials of degree greater than two on rational sectors. In addition, the conjecture of Nathanson [3] that there are no packing polynomials on $S(\alpha)$ for irrational α remains open.

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